The sine-Gordon model in the presence of defects

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Abstract

The sine-Gordon model in the presence of dynamical integrable defects is investigated. This is an application of the algebraic formulation introduced for integrable defects in earlier works. The quantities in involution as well as the associated Lax pairs are explicitly extracted. Integrability i also shown using certain sewing constraints, which emerge as suitable continuity conditions.

1 Introduction

The investigation of integrable defects has been a quite challenging problem, and there is a wealth of relevant articles in recent years at both classical and quantum level [1]–[18]. A fully algebraic picture for a description of a Liouville integrable defect was recently proposed [1]. It was exemplified in the case of the continuous non-linear Schrödinger model (NLS) [1], and now the procedure is extend to the sine Gordon model [2].

  This procedure will now be applied to the sine-Gordon (SG) model, for which we shall consider a generic dynamical defect (type II) (see also [4, 9]). We must immediately emphasize that this model provides an example where the initial off-shell continuous "conserved" Hamiltonians do not Poisson commute, but will be shown to weakly Poisson-commute once the sewing conditions are implemented, thereby guaranteeing Liouville integrability of the reduced model. By contrast in the NLS case the continuous Hamiltonians strongly (i.e. off-shell) Poisson-commuted. As seen above this does not modify the conclusions on Liouville integrability on-shell.

  This paper is mainly based on the analysis of [2], therefore for more details on the explicit algebraic formulation we refer the interested reader to the aforementioned relevant article.

2 General setting

The key point in the description of classical integrable lattice models is the existence of the Lax pair $\mathbb{U}$, $\mathbb{V}$. Define Ψ as being a solution of the following set of equations (see e.g. [20])

$$\frac{\partial \Psi}{\partial x} = \mathbb{U}(x, t, \lambda)\Psi \tag{1}$$

$$\frac{\partial \Psi}{\partial t} = \mathbb{V}(x, t, \lambda)\Psi \tag{2}$$

$\mathbb{U}$, $\mathbb{V}$ being in general $n \times n$ matrices with entries defined as functions of complex valued dynamical fields, their derivatives, and the complex spectral parameter λ. Compatibility of the two aforementioned equations (1), (2) gives rise to the zero curvature condition

$$\dot{\mathbb{U}}(x, t) - \mathbb{V}'(x, t) + \left[\mathbb{U}(x, t),\ \mathbb{V}(x, t)\right] = 0, \tag{3}$$

which provides the equations of motion of the system at hand.

As is well known the generating function of the local integrals of motion is given by the expression

$$\mathcal{G} = \ln(tr T(L, -L, \lambda)), \tag{4}$$

where the monodromy matrix T is defined as,

$$T(L, -L, \lambda) = P \exp\left\{ \int_{-L}^{L} dx \; \mathbb{U}(x) \right\}, \tag{5}$$

We now impose that the operator $\mathbb{U}$ satisfy the ultra-local Poisson structure described by the linear algebraic relations

$$\left\{ \mathbb{U}_1(x, \lambda), \; \mathbb{U}_2(y, \mu) \right\} = \left[r_{12}(\lambda - \mu), \; \mathbb{U}(x, \lambda) + \mathbb{U}_2(y, \mu) \right] \delta(x - y) \tag{6}$$

It is then straightforward to show that T satisfies the fundamental quadratic algebra:

$$\left\{ T_1(\lambda), \; T_2(\mu) \right\} = \left[r_{12}(\lambda - \mu), \; T_1(\lambda) \, T_2(\mu) \right]. \tag{7}$$

$r_{12}(\lambda - \mu)$ is the so-called classical r-matrix assumed here to be a non-dynamical skew-symmetric solution of the classical Yang-Baxter equation.

Let us now particularize our analysis to the sine-Gordon model. In this case the $\mathbb{U}$ operator of the Lax pair is a 2×2 matrix and is given by [20]:

$$\mathbb{U}(x, t, u) = \frac{\beta}{4i} \pi(x, t) \sigma^z + \frac{mu}{4i} e^{\frac{i\beta}{4}\phi\sigma^z} \sigma^y e^{-\frac{i\beta}{4}\phi\sigma^z} - \frac{mu^{-1}}{4i} e^{-\frac{i\beta}{4}\phi\sigma^z} \sigma^y e^{\frac{i\beta}{4}\phi\sigma^z} \tag{8}$$

$u \equiv e^{\lambda}$, $\sigma^{x,y,z}$ are the 2×2 Pauli matrices, and the associated classical r-matrix in this case is given by the familiar form [20]:

$$r(\lambda) = \frac{\beta^2}{8 \sinh \lambda} \begin{pmatrix} \frac{\sigma^z + 1}{2} \cosh \lambda & \sigma^- \\ \sigma^+ & \frac{-\sigma^z + 1}{2} \cosh \lambda \end{pmatrix}. \tag{9}$$

Stating that the Lax operator $\mathbb{U}$ satisfies the linear Poisson algebra (6) is equivalent to setting that ϕ, π are canonical conjugates, i.e.

$$\left\{ \phi(x), \; \pi(y) \right\} = \delta(x - y). \tag{10}$$

Let us now apply the generic defect construction to the sine-Gordon model.

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3 The sine-Gordon model with defect

The main ingredient in the description of integrable defects is the modified monodromy matrix. More precisely, in the presence of an integrable defect the monodromy matrix of the field theory is modified (see also [6, 11, 1]), and takes the generic form

$$
\begin{aligned}
T(L, -L, \lambda) &= T^+(L, x_0^+, \lambda)\ \tilde{L}(\lambda)\ T^-(x_0^-, -L, \lambda) \\
&= P \exp\left\{ \int_{x_0^+}^{L} dx\ \mathbb{U}^+(x) \right\} \tilde{L}(\lambda)\ P \exp\left\{ \int_{-L}^{x_0^-} dx\ \mathbb{U}^-(x) \right\} \quad (11)
\end{aligned}
$$

Assuming that the defect Lax matrix $\tilde{L}$ also satisfies the quadratic Poisson algebra (7) T given in (11) satisfies (7).

A consistent generic parametrization of an integrable defect of the so-called Type-II or dynamical will be considered. We assume the following form for the classical $\tilde{L}$ matrix

$$
\tilde{L}(\lambda) = \begin{pmatrix} e^{\lambda}V - e^{-\lambda}V^{-1} & \bar{a} \\ a & e^{\lambda}V^{-1} - e^{-\lambda}V \end{pmatrix}. \quad (12)
$$

Requiring that $\tilde{L}$ satisfies the algebraic relation (7), one then extracts the following Poisson relations between the defect fields:

$$
\begin{aligned}
\left\{ V,\ \bar{a} \right\} &= \frac{\beta^2}{8} V\ \bar{a}, \\
\left\{ V,\ a \right\} &= -\frac{\beta^2}{8} V a, \\
\left\{ \bar{a},\ a \right\} &= \frac{\beta^2}{4}(V^2 - V^{-2}) \quad (13)
\end{aligned}
$$

From these Poisson brackets one naturally extracts a cyclic variable $C_0 = V^2 + V^{-2} + \bar{a}a$ identified as the Casimir of a deformed $\mathfrak{sl}_2$. This variable Poisson-commute with all other dynamical quantities and can therefore be fixed to some particular value c_0. We shall nevertheless keep the redundant three-parameter expression for $\tilde{L}$ for reasons of form simplicity in the explicit expressions.

We shall express the term of order u in $\mathbb{U}$ independently of the fields, after applying a suitable gauge transformation [20]

$$
T^\pm(x, y, \lambda) = \Omega^\pm(x)\ \tilde{T}^\pm(x, y)\ (\Omega^\pm(y))^{-1}, \qquad \Omega^\pm = e^{\frac{i\beta}{4}\phi^\pm \sigma^z}, \quad (14)
$$

The gauge transformed operator $\tilde{\mathbb{U}}$ is expressed as:

$$\tilde{\mathbb{U}}^\pm(x,t,u) = \frac{\beta}{4i}\mathfrak{f}^\pm\sigma^z + \frac{mu}{4i}\sigma^y - \frac{mu^{-1}}{4i}e^{-\frac{i\beta}{2}\phi^\pm\sigma^z}\sigma^y e^{\frac{i\beta}{2}\phi^\pm\sigma^z} \tag{15}$$

where we define

$$\mathfrak{f}^\pm(x,t) = \pi^\pm(x,t) + \phi^{\pm\prime}(x,t). \tag{16}$$

We consider the following convenient decomposition for $\tilde{T}$, as $|u| \to \infty$ [20],

$$\tilde{T}^\pm(x,y,\lambda) = (1 + W^\pm(x,\lambda))\ e^{Z^\pm(x,y,\lambda)}\ (1 + W^\pm(y,\lambda))^{-1} \tag{17}$$

$W^\pm$ is an off-diagonal matrix and $Z^\pm$ is purely diagonal. They are expanded as:

$$W^\pm = \sum_{k=0}^{\infty} \frac{W^{\pm(k)}}{u^k}, \qquad Z^\pm = \sum_{k=-1}^{\infty} \frac{Z^{\pm(k)}}{u^k} \tag{18}$$

Note that $\tilde{T}$ naturally satisfies the gauged Lax equation:

$$\frac{\partial \tilde{T}^\pm}{\partial x} = \tilde{\mathbb{U}}^\pm(x,\lambda)\tilde{T}^\pm(x,y,\lambda) \tag{19}$$

Inserting expressions (17), (18) in (19) one identifies the matrices $W^{\pm(k)}$ and $Z^{\pm(k)}$. More precisely, we end up with an equation for the off-diagonal matrix:

$$\frac{\partial W^\pm}{\partial x} + W^\pm\tilde{\mathbb{U}}_D^\pm - \tilde{\mathbb{U}}_D^\pm W^\pm + W^\pm\tilde{\mathbb{U}}_A^\pm W^\pm - \mathbb{U}_A^\pm = 0 \tag{20}$$

where the indices D, A denote the diagonal and anti-diagonal part of the Lax operator $\tilde{\mathbb{U}}^\pm$, leading to typical Riccati-type equations for the entries of $W^\pm$ [2].

Similarities with corresponding equations emerging in [12, 18] from the inverse scattering point of view are apparent as expected, given that one essentially solves the same fundamental equations. Liouville integrability is guaranteed within the present approach by construction, at least formally, whereas in the methodology of [12, 18] only the conservation of the charges for a singled-out time-evolution is shown through the zero curvature condition i.e. explicit use of the equations of motion.

It is sufficient for our purposes here to identify only the first few terms of the expansions. Indeed based on equation (19) we conclude (see also [20]):

$$W^{\pm(0)} = i\sigma_1, \quad W^{\pm(1)} = -\frac{i\beta}{m}\mathfrak{f}^{\pm}(x)\sigma_1,$$

$$W^{\pm(2)} = \frac{2i\beta\mathfrak{f}^{\pm'}}{m^2}\sigma_2 - i\sin(\beta\phi^{\pm})\,\sigma_2 - \frac{\beta^2(\mathfrak{f}^{\pm})^2}{2im^2}\sigma_1. \quad (21)$$

We also need to identify the diagonal elements $Z^{\pm(n)}$. In particular from equation (1) we extract the following expressions:

$$Z^{+(-1)} = -\frac{im(L-x_0)}{4}\sigma_3, \quad Z^{-(-1)} = -\frac{im(L+x_0)}{4}\sigma_3$$

$$Z^{+(1)} = \frac{m}{4}\begin{pmatrix} -\int_{x_0^+}^{L} dx\, W_{21}^{+(2)}(x) & \\ & \int_{x_0^+}^{L} dx\, W_{12}^{+(2)}(x) \end{pmatrix}$$

$$-\frac{m}{4}\begin{pmatrix} -i\int_{x_0}^{L} dx\, e^{-i\beta\phi^+} & \\ & i\int_{x_0^+}^{L} dx\, e^{i\beta\phi^+} \end{pmatrix},$$

$$Z^{-(1)} = \frac{m}{4}\begin{pmatrix} -\int_{-L}^{x_0^-} dx\, W_{21}^{-(2)}(x) & \\ & \int_{-L}^{x_0^-} dx\, W_{12}^{-(2)}(x) \end{pmatrix}$$

$$-\frac{m}{4}\begin{pmatrix} -i\int_{-L}^{x_0^-} dx\, e^{-i\beta\phi^-} & \\ & i\int_{-L}^{x_0^-} dx\, e^{i\beta\phi^-} \end{pmatrix}. \quad (22)$$

Notice that for $-iu \to \infty$ the leading contribution comes from the $Z_{11}^{\pm}$ elements. This observation will be subsequently quite useful.

We shall first derive the associated local integrals of motion. In particular, the energy and momentum in the presence of defect will be obtained. Let us first recall the generating function of the local integrals of motion

$$\mathcal{G}(\lambda) = \ln\left[trT^+(L, x_0, \lambda)\,\tilde{L}(x_0, \lambda)\,T^-(x_0, L, \lambda)\right] \quad (23)$$

Schwartz boundary conditions are imposed at the end point of the system $\pm L$. Recalling also the ansatz for the monodromy matrices we conclude that the generating function:

$$\mathcal{G}(\lambda) = \ln tr\left[e^{Z^+(L,x_0)}(1 + W^+(x_0))^{-1}(\Omega^+(x_0))^{-1}\tilde{L}(x_0)\Omega^-(x_0)(1 + W^-(x_0))e^{Z^-(x_0,-L)}\right] \quad (24)$$

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Choosing to consider the $-iu \to \infty$ behavior we take into account the leading contribution for the $Z_{11}^{\pm}$ terms, then the generating function takes the form reads as:

$$\mathcal{G}(\lambda) = Z_{11}^{+} + Z_{11}^{-} + \ln \left[(1 + W^{+}(x_0))^{-1} (\Omega^{+}(x_0))^{-1} \tilde{L}(x_0) \Omega^{-}(x_0)(1 + W^{-}(x_0)) \right]_{11} (25)$$

Expanding the latter expression in powers of u^{-1} we obtain the following:

$$\mathcal{G}(\lambda) = \sum_{m=0}^{\infty} \frac{I^{(m)}}{u^m}. \tag{26}$$

Recalling now the expression for the generating function of integrals of motion we conclude that

$$
\begin{aligned}
I^{(1)} &= -\frac{m}{4i} \int_{-L}^{x_0^-} dx \left(-\frac{\beta^2}{2m^2} \mathfrak{f}^{-2}(x) + \cos(\beta \phi^-(x)) \right) \\
&- \frac{m}{4i} \int_{x_0^+}^{L} dx \left(-\frac{\beta^2}{2m^2} \mathfrak{f}^{+2}(x) + \cos(\beta \phi^+(x)) \right) \\
&+ \frac{i}{\mathcal{D}} \left(e^{-\frac{i\beta}{4}(\phi^+(x_0)+\phi^-(x_0))} \bar{a} - e^{\frac{i\beta}{4}(\phi^+(x_0)+\phi^-(x_0))} a \right) + \frac{\beta}{2m\mathcal{D}} \left(\mathfrak{f}^+(x_0) + \mathfrak{f}^-(x_0) \right) \mathcal{A}
\end{aligned}
$$

where we define:

$$
\begin{aligned}
\mathcal{D} &= e^{-\frac{i\beta}{4}(\phi^+(x_0)-\phi^-(x_0))} V + e^{\frac{i\beta}{4}(\phi^+(x_0)-\phi^-(x_0))} V^{-1}, \\
\mathcal{A} &= e^{-\frac{i\beta}{4}(\phi^+(x_0)-\phi^-(x_0))} V - e^{\frac{i\beta}{4}(\phi^+(x_0)-\phi^-(x_0))} V^{-1}. \tag{27}
\end{aligned}
$$

If we now perform the same expansion for $\lambda \to -\infty$, we basically end up with a similar expression, by simply exploiting the fundamental symmetry of the monodromy matrix:

$$T(u^{-1}, \phi, \ \pi, V, a, \bar{a}) = T(-u, \ -\phi, \ \pi, V^{-1}, a, \bar{a}). \tag{28}$$

More precisely, one concludes that:

$$
\begin{aligned}
I^{(-1)} &= \frac{m}{4i} \int_{-L}^{x_0^-} dx \left(-\frac{\beta^2}{2m^2} \hat{\mathfrak{f}}^{-2}(x) + \cos(\beta \phi^-(x)) \right) \\
&- \frac{m}{4i} \int_{x_0^+}^{L} dx \left(-\frac{\beta^2}{2m^2} \hat{\mathfrak{f}}^{+2}(x) + \cos(\beta \phi^+(x)) \right) \\
&- \frac{i}{\mathcal{D}} \left(e^{\frac{i\beta}{4}(\phi^+(x_0)+\phi^-(x_0))} \bar{a} - e^{-\frac{i\beta}{4}(\phi^+(x_0)+\phi^-(x_0))} a \right) + \frac{\beta}{2m\mathcal{D}} \left(\hat{\mathfrak{f}}^+(x_0) + \hat{\mathfrak{f}}^-(x_0) \right) \mathcal{A}
\end{aligned}
$$

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where we define

$$\hat{\mathfrak{f}}^{\pm}(\phi, \ \pi) = \mathfrak{f}^{\pm}(-\phi, \ \pi). \tag{29}$$

Any combination of the quantities $I^{(1)}$, $I^{(-1)}$ can be picked as one of the charges in involution. In particular the defect-extended form of the sine-Gordon Hamiltonian is defined as:

$$
\begin{aligned}
\mathcal{H} &= \frac{2im}{\beta^2}(I^{(1)} - I^{(-1)}) \\
&= \int_{-L}^{x_0^-} dx \ \left(\frac{1}{2}(\pi^{-2}(x) + \phi^{-'2}(x)) - \frac{m^2}{\beta^2}\cos(\beta\phi^-(x))\right) \\
&\quad + \int_{x_0^+}^{L} dx \ \left(\frac{1}{2}(\pi^{+2}(x) + \phi^{+'2}(x)) - \frac{m^2}{\beta^2}\cos(\beta\phi^+(x))\right) \\
&\quad - \frac{4m}{\beta^2\mathcal{D}}\cos\frac{\beta}{4}(\phi^+(x_0) + \phi^-(x_0))\left(\bar{a} - a\right) + \frac{2i}{\beta\mathcal{D}}\left(\phi^{+'}(x_0) + \phi^{-'}(x_0)\right)\mathcal{A}
\end{aligned}
$$

and we also identify the sine-Gordon momentum as:

$$
\begin{aligned}
\mathcal{P} &= \frac{2im}{\beta^2}\left(I^{(1)} + I^{(-1)}\right) \\
&= \int_{-L}^{x_0^-} dx \ \phi^{-'}(x)\pi^-(x) + \int_{x_0^+}^{L} dx \ \phi^{+'}(x)\pi^+(x) \\
&\quad + \frac{4mi}{\beta^2\mathcal{D}}\sin\frac{\beta}{4}(\phi^+(x_0) + \phi^-(x_0))\left(\bar{a} + a\right) + \frac{2i}{\beta\mathcal{D}}\left(\pi^+(x_0) + \pi^-(x_0)\right)\mathcal{A}.
\end{aligned}
$$

Explicit computation of the Poisson bracket $\{\mathcal{H}, \mathcal{P}\}$ now yields a number of non-zero terms; we shall come back to this issue after deriving the sewing conditions.

The next step is the derivation of the time components of the associated Lax pairs. Expressions of the time component $\mathbb{V}$ of the Lax pair are known (see e.g. [21, 20]). The generic expressions for the bulk left and right theories as well as the defect points are given as [1, 22]:

$$
\begin{aligned}
\mathbb{V}^+(x, \lambda, \mu) &= t^{-1}(\lambda)tr_a\Big(T_a^+(L, x, \lambda)r_{ab}(\lambda - \mu)T_a^+(x, x_0, \lambda)\tilde{L}_a(x_0, \lambda)T_a^-(x_0, -L, \lambda)\Big) \\
\mathbb{V}^-(x, \lambda, \mu) &= t^{-1}(\lambda)tr_a\Big(T_a^+(L, x_0, \lambda)\tilde{L}_a(x_0)T_a^-(x_0, x, \lambda)r_{ab}(\lambda - \mu)T_a^-(x, -L, \lambda)\Big) \\
\tilde{\mathbb{V}}^+(x_0, \lambda, \mu) &= t^{-1}(\lambda)tr_a\Big(T_a^+(L, x_0, \lambda)r_{ab}(\lambda - \mu)\tilde{L}_a(x_0, \lambda)T_a^-(x_0, -L, \lambda)\Big) \\
\tilde{\mathbb{V}}^-(x_0, \lambda, \mu) &= t^{-1}(\lambda)tr_a\Big(T_a^+(L, x_0, \lambda)\tilde{L}_a(x_0, \lambda)r_{ab}(\lambda - \mu)T_a^-(x_0, -L, \lambda)\Big). \tag{30}
\end{aligned}
$$

In order to identify the Lax pair associated to the Hamiltonian and momentum it is necessary to formulate the expansion of $\mathbb{V}$ in both negative and positive powers of u.

The first order contribution in the u^{-1} expansion of the bulk $\mathbb{V}^{\pm}$ operator (we have set the second spectral parameter $v \equiv e^{\mu}$) reads:

$$\mathbb{V}^{\pm(1)} = \frac{\beta^2}{8}\left(\frac{\beta}{2m}\sigma^z(\pi^{\pm} + \phi^{\pm'}) + iv\left(\sigma^- e^{-\frac{i\beta}{2}\phi^{\pm}} - \sigma^+ e^{\frac{i\beta}{2}\phi^{\pm}}\right)\right) \qquad (31)$$

The first order contribution in the u expansion reads:

$$\hat{\mathbb{V}}^{\pm(1)} = \frac{\beta^2}{8}\left(\frac{\beta}{2m}\sigma^z(\pi^{\pm} - \phi^{\pm'}) - iv^{-1}\left(\sigma^- e^{\frac{i\beta}{2}\phi^{\pm}} - \sigma^+ e^{-\frac{i\beta}{2}\phi^{\pm}}\right)\right) \qquad (32)$$

Subtracting these two expressions and multiplying by $-\frac{2im}{\beta^2}$ we obtain the time component of the Lax pair associated to the Hamiltonian: ($\Omega^{\pm} = e^{\frac{i\beta}{4}\phi^{\pm}\sigma^z}$)

$$\mathbb{V}_{\mathcal{H}}^{\pm} = \frac{\beta}{4i}\phi^{\pm'}\sigma^z + \frac{vm}{4i}\Omega^{\pm}\sigma^y(\Omega^{\pm})^{-1} + \frac{v^{-1}m}{4i}(\Omega^{\pm})^{-1}\sigma^y\Omega^{\pm} \qquad (33)$$

Adding now (31), (32), after multiplying with $-\frac{2im}{\beta^2}$, provides the time component of the Lax pair associated to the momentum:

$$\mathbb{V}_{\mathcal{P}}^{\pm} = \frac{\beta}{4i}\pi^{\pm}\sigma^z + \frac{vm}{4i}\Omega^{\pm}\sigma^y(\Omega^{\pm})^{-1} - \frac{v^{-1}m}{4i}(\Omega^{\pm})^{-1}\sigma^y\Omega^{\pm} \qquad (34)$$

The next step is the derivation of the relevant Lax pairs for the defect point from the left and the right, based on the expression (30). Indeed, after some cumbersome but quite straightforward computations, and after defining:

$$w^{\pm} = -\frac{i\beta}{m}\mathfrak{f}^{\pm}, \qquad \hat{w}^{\pm} = \frac{i\beta}{m}\hat{\mathfrak{f}}^{\pm}, \qquad (35)$$

we conclude from the expansion in powers of u^{-1}:

$$\begin{aligned}
\tilde{\mathbb{V}}^{+(1)} &= \frac{i\beta^2}{8}\mathcal{D}^{-2}\sigma^z\left[w^+ + w^- + e^{\frac{i\beta}{2}\phi^-}Va + e^{-\frac{i\beta}{2}\phi^-}V^{-1}\bar{a}\right] \\
&+ \frac{i\beta^2}{4}\mathcal{D}^{-1}v\left[\sigma^- e^{-\frac{i\beta}{4}(\phi^+ + \phi^-)}V^{-1} - \sigma^+ e^{\frac{i\beta}{4}(\phi^+ + \phi^-)}V\right], \qquad (36)
\end{aligned}$$

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whereas the expansion in powers of u leads to:

$$\hat{\tilde{\mathbb{V}}}^{+(1)} = -\frac{i\beta^2}{8}\mathcal{D}^{-2}\sigma^z\Big[\hat{w}^+ + \hat{w}^- - e^{\frac{i\beta}{2}\phi^-}V\bar{a} - e^{-\frac{i\beta}{2}\phi^-}V^{-1}a\Big]$$
$$- \frac{i\beta^2}{4}\mathcal{D}^{-1}v^{-1}\Big[\sigma^- e^{\frac{i\beta}{4}(\phi^++\phi^-)}V - \sigma^+ e^{-\frac{i\beta}{4}(\phi^++\phi^-)}V^{-1}\Big] \quad (37)$$

Similarly, the corresponding expressions for $\tilde{\mathbb{V}}^{-(1)}$, $\tilde{\mathbb{V}}^{-(1)}$ are given below:

$$\tilde{\mathbb{V}}^{-(1)} = \frac{i\beta^2}{8}\mathcal{D}^{-2}\sigma^z\Big[w^+ + w^- - e^{\frac{i\beta}{2}\phi^+}V^{-1}a - e^{-\frac{i\beta}{2}\phi^+}V\bar{a}\Big]$$
$$+ \frac{i\beta^2}{4}\mathcal{D}^{-1}v\Big[\sigma^- e^{-\frac{i\beta}{4}(\phi^++\phi^-)}V - \sigma^+ e^{\frac{i\beta}{4}(\phi^++\phi^-)}V^{-1}\Big] \quad (38)$$

$$\hat{\tilde{\mathbb{V}}}^{-(1)} = -\frac{i\beta^2}{8}\mathcal{D}^{-2}\sigma^z\Big[\hat{w}^+ + \hat{w}^- + e^{\frac{i\beta}{2}\phi^+}V^{-1}\bar{a} + e^{-\frac{i\beta}{2}\phi^+}Va\Big]$$
$$- \frac{i\beta^2}{4}\mathcal{D}^{-1}v^{-1}\Big[\sigma^- e^{\frac{i\beta}{4}(\phi^++\phi^-)}V^{-1} - \sigma^+ e^{-\frac{i\beta}{4}(\phi^++\phi^-)}V\Big]. \quad (39)$$

We are now in a position to apply the scheme elaborated in [1]. The first manifest observation from the continuity conditions

$$\tilde{\mathbb{V}}^{+(1)}(x_0) \to \mathbb{V}^{+(1)}(x_0^+), \qquad x_0^+ \to x_0$$
$$\tilde{\mathbb{V}}^{-(1)}(x_0) \to \mathbb{V}^{-(1)}(x_0^-), \qquad x_0^- \to x_0 \qquad (40)$$

(similar continuity conditions apply for the "hatted" quantities, but are omitted for brevity), is that:

$$V = e^{\frac{i\beta}{4}(\phi^+-\phi^-)}. \qquad (41)$$

and will be hereafter denoted as "first sewing condition S_1 ". Remember that from the very beginning one has already fixed the Casimir C_0 to some value c_0 independently of any sewing requirement. This can be seen as an "order zero condition S_0 " without any dependance in the bulk variables and yields a first-class constraint Poisson-commuting with all dynamical variables.

After imposing (41) the time components of the Lax pairs on the defect point take the following simple expressions:

$$\tilde{\mathbb{V}}^{\pm(1)} = \frac{\beta^2}{8}\left(\frac{\beta}{4m}\sigma^z\Big(\pi^+ + \phi^{+\prime} + \pi^- + \phi^{-\prime}\Big) \pm \frac{i\sigma^z}{4}\mathbb{M} + iv\Big(\sigma^- e^{-\frac{i\beta}{2}\phi^\pm} - \sigma^+ e^{\frac{i\beta}{2}\phi^\pm}\Big)\right)(42)$$

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and the first term in the u expansion provides:

$$\hat{\mathbb{V}}^{\pm} = \frac{\beta^2}{8}\left(\frac{\beta}{4m}\sigma^z\left(\pi^+ - \phi^{+'} + \pi^- - \phi^{-'}\right) \pm \frac{i\sigma^z}{4}\hat{\mathbb{M}} - iv^{-1}\left(\sigma^- e^{\frac{i\beta}{2}\phi^{\pm}} - \sigma^+ e^{-\frac{i\beta}{2}\phi^{\pm}}\right)\right)(43)$$

where we define:

$$\mathbb{M} = e^{-\frac{i\beta}{4}(\phi^+ +\phi^-)}\bar{a} + e^{\frac{i\beta}{4}(\phi^+ +\phi^-)}a$$
$$\hat{\mathbb{M}} = e^{\frac{i\beta}{4}(\phi^+ +\phi^-)}\bar{a} + e^{-\frac{i\beta}{4}(\phi^+ +\phi^-)}a \qquad (44)$$

Continuity conditions on the Lax pair as also described in (40) give rise to the following sewing conditions on the defect point x_0 associated to the momentum and the Hamiltonian respectively:

$$S_2: \qquad \pi^+(x_0) - \pi^-(x_0) = \frac{im}{\beta}\cos\frac{\beta}{4}(\phi^+(x_0) + \phi^-(x_0))\left(a + \bar{a}\right)$$

$$S_2': \qquad \phi^{+'}(x_0) - \phi^{-'}(x_0) = \frac{m}{\beta}\sin\frac{\beta}{4}(\phi^+(x_0) + \phi^-(x_0))\left(\bar{a} - a\right) \ (45)$$

the prime denotes the derivative with respect to x.

It is instructive to point out that comparison of the extracted charges (30), (30), and the latter equations (45) with similar results obtained for instance in [11, 12] reveal manifest discrepancies. We shall further comment on this matter in the discussion section.

Consistency of the sewing conditions S_1, S_2, S_2' can now be checked by computing their Poisson brackets with the first two Hamiltonians $\mathcal{H}, \mathcal{P}$. Indeed one gets:

$$\left\{\mathcal{H}, S_1\right\} = -\frac{i\beta}{4}(\pi^+(x_0) - \pi^-(x_0))S_1 \ + \ \frac{i\beta}{4}S_2V + o(\mathcal{D} - 2)$$

$$\left\{\mathcal{P}, S_1\right\} = -\frac{i\beta}{4}(\phi^{+'}(x_0) - \phi^{-'}(x_0))S_1 \ + \ \frac{i\beta}{4}S_2'V + o(\mathcal{D} - 2) \quad (46)$$

We recall that on-shell $\mathcal{D} \approx 2$; $\mathcal{A} \approx 0$.

Consider now the Poisson brackets of $\mathcal{H}$, $\mathcal{P}$ with S_2, S_2'. One easily obtains that they are given by expressions of the following form:

$$\left\{\mathcal{P}, S_2\right\} = (\pi^{+'}(x_0) - \pi^{-'}(x_0)) \ F(\pi^+(x_0) + \pi^-(x_0), \phi^{+'}(x_0) + \phi^{-'}(x_0), \phi^+(x_0), \phi^-(x_0), V, a, \bar{a})$$

$$\left\{\mathcal{H}, S_2\right\} = (\phi^{+''}(x_0) - \phi^{-''}(x_0)) \ G(\pi^+(x_0) + \pi^-(x_0), \phi^{+'}(x_0) + \phi^{-'}(x_0), \phi^+(x_0), \phi^-(x_0), V, a, \bar{a})$$

$$(47)$$

where F and G are given functions to be computed specifically. Poisson brackets with S_2' are given by similar expressions exchanging $\mathcal{H}$ and $\mathcal{P}$. Note that (contrary to the non-linear Schroedinger case) no term proportional to the singular contribution $\delta(0)$ arise, they fully cancel in the Poisson brackets. It is therefore to be expected that the finite terms on the r.h.s. of both PB's will yield the third sewing conditions S_3, S_3' which will respectively take the form (expected from general arguments) $(\pi^{+'}(x_0) - \pi^{-'}(x_0)) = -F$ and $(\phi^{+''}(x_0) - \phi^{-''}(x_0) = -G$. Explicit derivation of these sewing conditions from higher terms in the expansion of the

$\mathbb{V}$ operators is technically quite cumbersome but we conjecture that they will coincide with the rhs of (47).

Let us now reconsider the Poisson bracket $\{\mathcal{H}, \mathcal{P}\}$. It turns out from explicit computations that one has in fact:

$$\left\{\mathcal{H}, \ \mathcal{P}\right\} \approx 0, \tag{48}$$

i.e. the Poisson bracket vanishes provided that the constraints $S_1, \ S_2, \ S_2'$ be satisfied.

Assuming that the Hamiltonians $\mathcal{H}$ and $\mathcal{P}$ weakly preserve all constraints (as already established for S_1 and conjectured for S_2, S_2') we deduce that the momentum and Hamiltonian Dirac commute. Our construction of Type II defect is thus compatible with a statement of Liouville-integrability on-shell. A general formal argument based on the underlying Poisson structure has developed in [1] establishing the property of weak preservation of constraints.

Equations of motion
To extract the associated equations of motion for the left and right bulk theories as well as the defect point one needs to employ the zero curvature condition expressed as:

$$\dot{\mathbb{U}}^{\pm}(x,t) - \mathbb{V}^{\pm'}(x,t) + \left[\mathbb{U}^{\pm}(x,t), \mathbb{V}^{\pm}(x,t)\right] = 0 \qquad x \neq x_0. \tag{49}$$

As usual the dot denotes derivative with respect to t.

On the defect point in particular the zero curvature condition is formulated as (this is also transparent when discussing the continuum limit of discrete theories (see e.g. [1])

$$\dot{\tilde{L}}(x_0) = \tilde{\mathbb{V}}^+(x_0)\tilde{L}(x_0) - \tilde{L}(x_0)\tilde{\mathbb{V}}^-(x_0), \tag{50}$$

and describes explicitly the jump occurring across the defect point.

The equations of motion are obtained via the zero curvature conditions as described above or (equivalently thanks to the sewing conditions) via the Hamiltonian equations i.e.

$$\dot{\phi}^{\pm} = \left\{\mathcal{H},\ \phi^{\pm}\right\}, \qquad \dot{\pi}^{\pm} = \left\{\mathcal{H},\ \pi^{\pm}\right\},$$
$$\dot{e} = \left\{\mathcal{H},\ e\right\}, \qquad e \in \left\{a,\ \bar{a},\ V\right\} \tag{51}$$

bear also in mind that

$$\left\{\pi^{\pm},\ e\right\} = \left\{\phi^{\pm},\ e\right\} = 0. \tag{52}$$

For the left and right bulk theories the familiar equations of motion for the sine-Gordon model arise

$$\ddot{\phi}^{\pm}(x,t) - \phi^{\pm\prime\prime}(x,t) + \frac{m^2}{\beta}\sin(\beta\phi^{\pm}(x,t)) = 0 \tag{53}$$

On the defect point the time evolution of the defect degrees of freedom are obtained as:

$$\dot{a} = -\frac{m}{2\mathcal{D}^2}\,\mathcal{A}\,a\,\cos\frac{\beta}{4}(\phi^+ + \phi^-)\left(\bar{a} - a\right) - \frac{m}{\mathcal{D}}\,\cos\frac{\beta}{4}(\phi^+ + \phi^-)\left(V^2 - V^{-2}\right)$$
$$- \frac{\beta i}{\mathcal{D}^2}\,a\left(\phi^{+\prime} + \phi^{-\prime}\right) \tag{54}$$

$$\dot{\bar{a}} = \frac{m}{2\mathcal{D}^2}\,\mathcal{A}\,\bar{a}\,\cos\frac{\beta}{4}(\phi^+ + \phi^-)\left(\bar{a} - a\right) - \frac{m}{\mathcal{D}}\,\cos\frac{\beta}{4}(\phi^+ + \phi^-)\left(V^2 - V^{-2}\right)$$
$$+ \frac{i\beta}{\mathcal{D}^2}\,\bar{a}\left(\phi^{+\prime} + \phi^{-\prime}\right) \tag{55}$$

$$\dot{V} = \frac{m}{2\mathcal{D}}\,V\,\cos(\frac{\beta}{4}(\phi^+ + \phi^-))\left(a + \bar{a}\right). \tag{56}$$

With this we conclude our presentation on the sine-Gordon model in the presence of integrable dynamical defects.

Acknowledgments

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